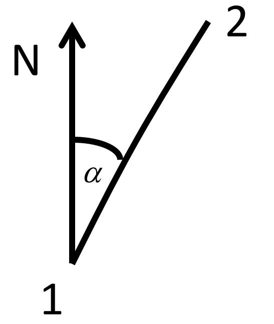
Observation Equations for Triangulation Computations

Observable quantities are function of unknowns



For azimuth,

$$\alpha = F(\varphi_1, \lambda_1, \varphi_2, \lambda 2) \quad - \quad \text{Common form}$$

Consider one of the above equation

$$\alpha_1 = F_1(\varphi_1, \lambda_1, \varphi_2, \lambda 2)$$

Let α_1' be the actual observed value and v_1 be the residual to the observed value

$$\alpha_1 = \alpha_1' + v_1$$

Also let, $\varphi_1^0, \lambda_1^0, \varphi_2^0, \lambda_2^0$ be the approximate values of the unknowns and $d\varphi_1, d\lambda_1, d\varphi_2, d\lambda_2$ be the corresponding corrections. Then

$$\alpha_1' + v_1 = F_1(\varphi_1^0 + d\varphi_1,\ \lambda_1^0 + d\lambda_1,\ \varphi_2^0 + d\varphi_2,\ \lambda_2^0 + d\lambda_2)$$

Expanding using Taylor's series

$$\alpha_1' + v_1 = F_1(\varphi_1^0,\ \lambda_1^0,\ \varphi_2^0,\ \lambda_2^0) + \left[\left(\frac{\partial F_1}{\partial \varphi_1}\right)_0 d\varphi_1 + \left(\frac{\partial F_1}{\partial \lambda_1}\right)_0 d\lambda_1 + \left(\frac{\partial F_1}{\partial \varphi_2}\right)_0 d\varphi_2 + \left(\frac{\partial F_1}{\partial \lambda_2}\right)_0 d\lambda_2 + \dots\right]$$

$$v_1 = \left(\alpha_1^0 - \alpha_1'\right) + \left(\frac{\partial F_1}{\partial \varphi_1}\right)_0 d\varphi_1 + \left(\frac{\partial F_1}{\partial \lambda_1}\right)_0 d\lambda_1 + \left(\frac{\partial F_1}{\partial \varphi_2}\right)_0 d\varphi_2 + \left(\frac{\partial F_1}{\partial \lambda_2}\right)_0 d\lambda_2$$

$$v_1 = \left(\alpha_1^0 - \alpha_1'\right) + d\alpha_{12t}$$

- **Observation equation for azimuth**

$$v_1 = \left(s_1^0 - s_1'\right) + ds_t$$

- Observation equation for distance

In matrix form,

Let $\left(\frac{\partial F_1}{\partial \varphi_1}\right)_0 = a_1$, $\left(\frac{\partial F_1}{\partial \lambda_1}\right)_0 = b_1$, $\left(\frac{\partial F_1}{\partial \varphi_2}\right)_0 = c_1$, $\left(\frac{\partial F_1}{\partial \lambda_2}\right)_0 = d_1$

Then $v_1 = a_1\, d\varphi_1 + b_1\, d\lambda_1 + c_1\, d\varphi_2 + d_1\, d\lambda_2 + f_1$ where $f_1 = \left(\alpha_1^0 - \alpha_1'\right)$

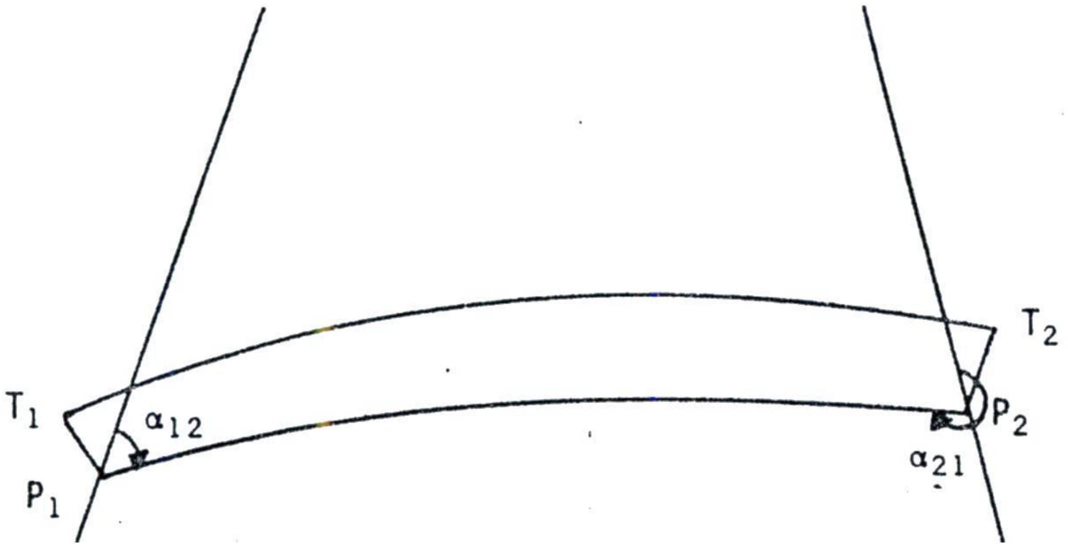
$$\begin{array}{c} . \\ . \end{array}$$

$$v_n = a_n\, d\varphi_1 + b_n\, d\lambda_1 + c_n\, d\varphi_2 + d_n\, d\lambda_2 + f_n$$

$$\begin{pmatrix} v_1 \\ v_2 \\ . \\ . \\ v_n \end{pmatrix}_{n\times 1} = \begin{pmatrix} a_1 & b_1 & c_1 & d_1 & \cdots \\ a_2 & b_2 & c_2 & d_2 & \cdots \\ . & & & & \\ . & & & & \\ a_n & b_n & c_n & d_n & \cdots \end{pmatrix}_{n\times u} \begin{pmatrix} d\varphi_1 \\ d\lambda_1 \\ d\varphi_2 \\ d\lambda_2 \\ . \end{pmatrix}_{u\times 1} + \begin{pmatrix} f_1 \\ f_2 \\ . \\ . \\ f_n \end{pmatrix}_{n\times 1}$$

Jacobian matrix (B matrix) F matrix

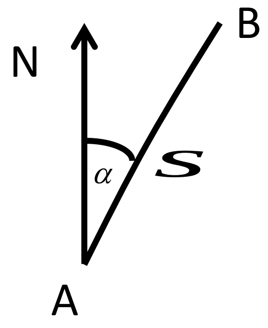
By considering the differential movements of the end points of the azimuth line, it can be shown that



$$d\alpha_{12t} = \frac{1}{s}\left[M_1 \sin\alpha_{12}.d\varphi_1 + M_2 \sin\alpha_{21}.d\varphi_2 - N_2 \cos\varphi_2 \cos\alpha_{21} (d\lambda_2 - d\lambda_1) \right]$$

$$ds_t = -M_2 \cos\alpha_{21}.d\varphi_2 - M_1 \cos\alpha_{12}.d\varphi_1 - N_2 \cos\varphi_2 \sin\alpha_{21} (d\lambda_2 - d\lambda_1)$$

Summary



- Observation equation for azimuth AB

$$v=\left(\alpha^0-\alpha'\right)+d\alpha_{ABt}$$

where, $d\alpha_{ABt}=\frac{1}{AB}\left[M_A\sin\alpha_{AB}.d\varphi_A+M_B\sin\alpha_{BA}.d\varphi_B-N_B\cos\varphi_B\cos\alpha_{BA}(d\lambda_B-d\lambda_A)\right]$

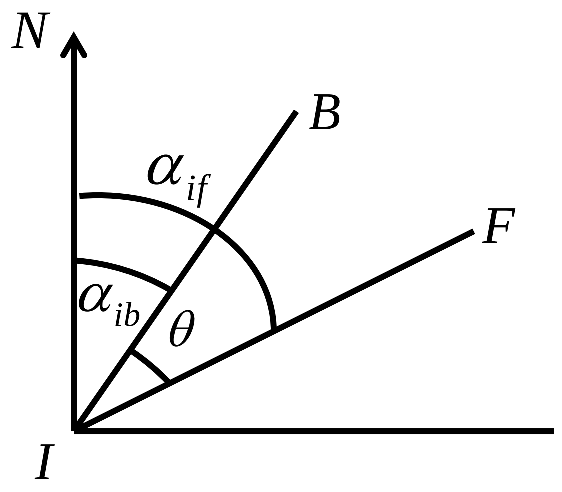
- For distance

$$v=\left(s^0-s'\right)+ds_t$$

where, $ds_t=-M_B\cos\alpha_{BA}.d\varphi_B-M_A\cos\alpha_{AB}.d\varphi_A-N_B\cos\varphi_B\sin\alpha_{BA}(d\lambda_B-d\lambda_A)$

In case that the observed quantities are included angles

$$\theta = \alpha_{if} - \alpha_{ib}$$



We know that

$$v_{if} = \left(\alpha_{if}^{0} - \alpha_{if}'\right) + d\alpha_{ift}$$

$$v_{ib} = \left(\alpha_{ib}^{0} - \alpha_{ib}'\right) + d\alpha_{ibt}$$

So, $$v_{if} - v_{ib} = (\alpha_{if}^{0} - \alpha_{ib}^{0}) - (\alpha_{if}' - \alpha_{ib}') + (d\alpha_{ift} - d\alpha_{ibt})$$

$$v_{\theta} = (\theta^{0} - \theta') + d\alpha_{\theta}$$

- Observation equation for included angles

where,
$$d\alpha_{\theta} = d\alpha_{ift} - d\alpha_{ibt}$$

$$= \frac{1}{IF}\left[M_i \sin\alpha_{if}.d\varphi_i + M_f \sin\alpha_{fi}.d\varphi_f - N_f \cos\varphi_f \cos\alpha_{fi}(d\lambda_f - d\lambda_i)\right] -$$

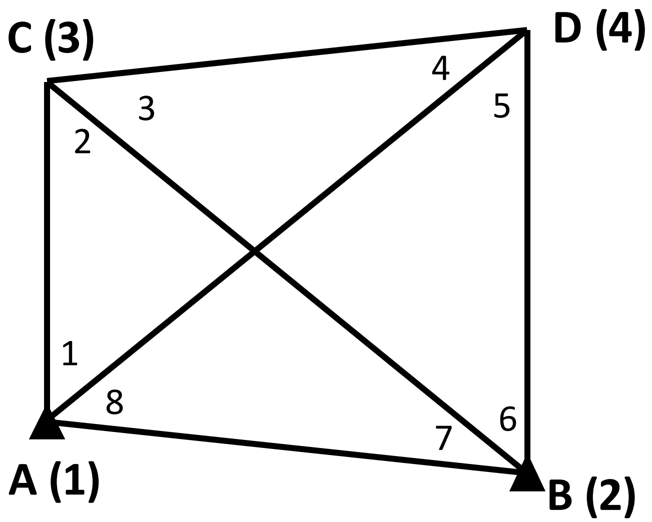
$$\frac{1}{IB}\left[M_i \sin\alpha_{ib}.d\varphi_i + M_b \sin\alpha_{bi}.d\varphi_b - N_b \cos\varphi_b \cos\alpha_{bi}(d\lambda_b - d\lambda_i)\right]$$

Example

1 and **2** are known stations.

N=8 (included angles)

U=4 (coordinates of 2 known points)



Eq.1 : $i=1,\ \ f=4,\ \ b=3$

$$v_1=\left(\frac{M_4}{AD}\sin\alpha_{41}\right)d\varphi_4-\left(\frac{M_3}{AC}\sin\alpha_{31}\right)d\varphi_3+\left(\frac{N_3}{AC}\cos\varphi_3\cos\alpha_{31}\right)d\lambda_3-\left(\frac{N_4}{AD}\cos\varphi_4\cos\alpha_{41}\right)d\lambda_4+\left(\theta_1^0-\theta_1'\right)\ ---(1)$$

Eq.2 : $i=3,\ \ f=1,\ \ b=2$

$$v_2=\left(\frac{M_3}{CA}\sin\alpha_{31}-\frac{M_3}{CB}\sin\alpha_{32}\right)d\varphi_3+\left(\frac{N_1}{CA}\cos\varphi_1\cos\alpha_{13}-\frac{N_2}{CB}\cos\varphi_2\cos\alpha_{23}\right)d\lambda_3+\left(\theta_2^0-\theta_2'\right)\ ---(2)$$

Similarly,

$$v_3 = \left(\frac{M_3}{CB} \sin\alpha_{32} - \frac{M_3}{CD} \sin\alpha_{34} \right) d\varphi_3 + \left(\frac{N_2}{CB} \cos\varphi_2 \cos\alpha_{23} - \frac{N_4}{CD} \cos\varphi_4 \cos\alpha_{43} \right) d\lambda_3 - \left(\frac{M_4}{CD} \sin\alpha_{43} \right) d\varphi_4 +$$

$$\left(\frac{N_4}{CD} \cos\varphi_4 \cos\alpha_{43} \right) d\lambda_4 + \left(\theta_3^0 - \theta_3' \right) \; ---(3)$$

$$v_4 = \left(\frac{M_3}{DC} \sin\alpha_{34} \right) d\varphi_3 - \left(\frac{N_3}{DC} \cos\varphi_3 \cos\alpha_{34} \right) d\lambda_3 + \left(\frac{M_4}{DC} \sin\alpha_{43} - \frac{M_4}{DA} \sin\alpha_{41} \right) d\varphi_4 +$$

$$\left(\frac{N_3}{DC} \cos\varphi_3 \cos\alpha_{34} - \frac{N_1}{DA} \cos\varphi_1 \cos\alpha_{14} \right) d\lambda_4 + \left(\theta_4^0 - \theta_4' \right) \; ---(4)$$

$$v_5 = \left(\frac{M_4}{DA} \sin\alpha_{41} - \frac{M_4}{DB} \sin\alpha_{42} \right) d\varphi_4 + \left(\frac{N_1}{DA} \cos\varphi_1 \cos\alpha_{14} - \frac{N_2}{DB} \cos\varphi_2 \cos\alpha_{24} \right) d\lambda_4 + \left(\theta_5^0 - \theta_5' \right) \; ---(5)$$

$$v_6 = \left(-\frac{M_3}{BC} \sin\alpha_{32} \right) d\varphi_3 + \left(\frac{N_3}{BC} \cos\varphi_3 \cos\alpha_{32} \right) d\lambda_3 + \left(\frac{M_4}{BD} \sin\alpha_{42} \right) d\varphi_4 - \left(\frac{N_4}{BD} \cos\varphi_4 \cos\alpha_{42} \right) d\lambda_4 +$$

$$\left(\theta_6^0 - \theta_6' \right) \; ---(6)$$

$$v_7 = \left(\frac{M_3}{BC} \sin\alpha_{32} \right) d\varphi_3 - \left(\frac{N_3}{BC} \cos\varphi_3 \cos\alpha_{32} \right) d\lambda_3 + \left(\theta_7^0 - \theta_7' \right) \; ---(7)$$

$$v_8 = \left(-\frac{M_4}{AD} \sin\alpha_{41} \right) d\varphi_4 + \left(\frac{N_4}{AD} \cos\varphi_4 \cos\alpha_{41} \right) d\lambda_4 + \left(\theta_8^0 - \theta_8' \right) \; ---(8)$$

In matrix form,

$$
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \end{bmatrix}_{8\times 1}
=
\begin{bmatrix}
\left(-\frac{M_3}{AC}\sin\alpha_{31}\right) & \left(\frac{N_3}{AC}\cos\varphi_3\cos\alpha_{31}\right) & \left(\frac{M_4}{AD}\sin\alpha_{41}\right) & \left(-\frac{N_4}{AD}\cos\varphi_4\cos\alpha_{41}\right) \\
\left(\frac{M_3}{CA}\sin\alpha_{31}-\frac{M_3}{CB}\sin\alpha_{32}\right) & \left(\frac{N_1}{CA}\cos\varphi_1\cos\alpha_{13}-\frac{N_2}{CB}\cos\varphi_2\cos\alpha_{23}\right) & 0 & 0 \\
\left(\frac{M_3}{CB}\sin\alpha_{32}-\frac{M_3}{CD}\sin\alpha_{34}\right) & \left(\frac{N_2}{CB}\cos\varphi_2\cos\alpha_{23}-\frac{N_4}{CD}\cos\varphi_4\cos\alpha_{43}\right) & -\left(\frac{M_4}{CD}\sin\alpha_{43}\right) & \left(\frac{N_4}{CD}\cos\varphi_4\cos\alpha_{43}\right) \\
\left(\frac{M_3}{DC}\sin\alpha_{34}\right) & -\left(\frac{N_3}{DC}\cos\varphi_3\cos\alpha_{34}\right) & \left(\frac{M_4}{DC}\sin\alpha_{43}-\frac{M_4}{DA}\sin\alpha_{41}\right) & \left(\frac{N_3}{DC}\cos\varphi_3\cos\alpha_{34}-\frac{N_1}{DA}\cos\varphi_1\cos\alpha_{14}\right) \\
0 & 0 & \left(\frac{M_4}{DA}\sin\alpha_{41}-\frac{M_4}{DB}\sin\alpha_{42}\right) & \left(\frac{N_1}{DA}\cos\varphi_1\cos\alpha_{14}-\frac{N_2}{DB}\cos\varphi_2\cos\alpha_{24}\right) \\
\left(-\frac{M_3}{BC}\sin\alpha_{32}\right) & \left(\frac{N_3}{BC}\cos\varphi_3\cos\alpha_{32}\right) & \left(\frac{M_4}{BD}\sin\alpha_{42}\right) & -\left(\frac{N_4}{BD}\cos\varphi_4\cos\alpha_{42}\right) \\
\left(\frac{M_3}{BC}\sin\alpha_{32}\right) & -\left(\frac{N_3}{BC}\cos\varphi_3\cos\alpha_{32}\right) & 0 & 0 \\
0 & 0 & \left(-\frac{M_4}{AD}\sin\alpha_{41}\right) & \left(\frac{N_4}{AD}\cos\varphi_4\cos\alpha_{41}\right)
\end{bmatrix}_{8\times 4}
\begin{bmatrix} d\varphi_3 \\ d\lambda_3 \\ d\varphi_4 \\ d\lambda_4 \end{bmatrix}_{4\times 1}
+
\begin{bmatrix} \theta_1^0-\theta_1' \\ \theta_2^0-\theta_2' \\ \theta_3^0-\theta_3' \\ \theta_4^0-\theta_4' \\ \theta_5^0-\theta_5' \\ \theta_6^0-\theta_6' \\ \theta_7^0-\theta_7' \\ \theta_8^0-\theta_8' \end{bmatrix}_{8\times 1}
$$

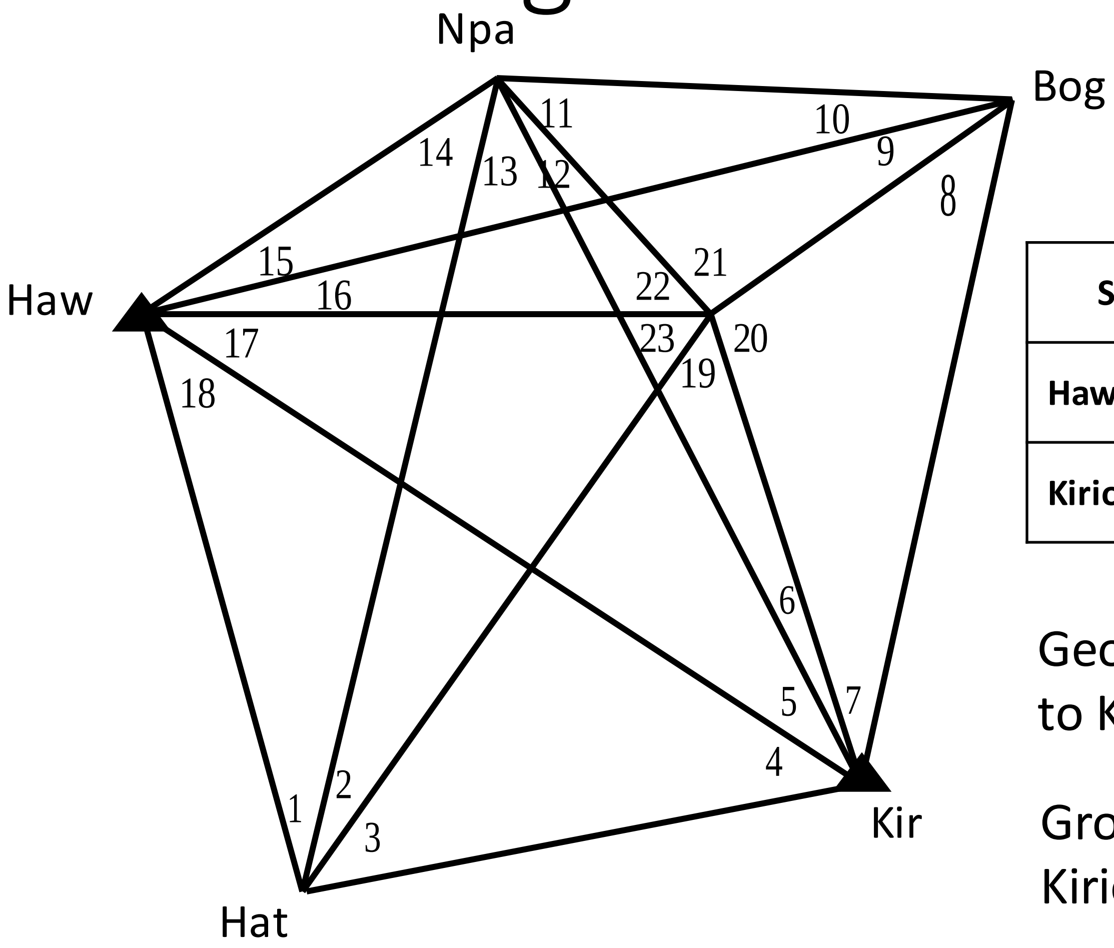
$$v = BX + F$$

Therefore, the least-squares solution of unknowns is

$$X = \left(B^T W B\right)^{-1} (B^T W F)$$

Then correct the included angles and determine the coordinates

Geodetic Triangulation Task-2014



Stn.	Latitude	Longitude
Hawagala	$6°43'8''.06866\ N$	$80°44'42.38747\ E$
Kirioluhena	$6°37'16''.66488\ N$	$80°49'58''.27704\ E$

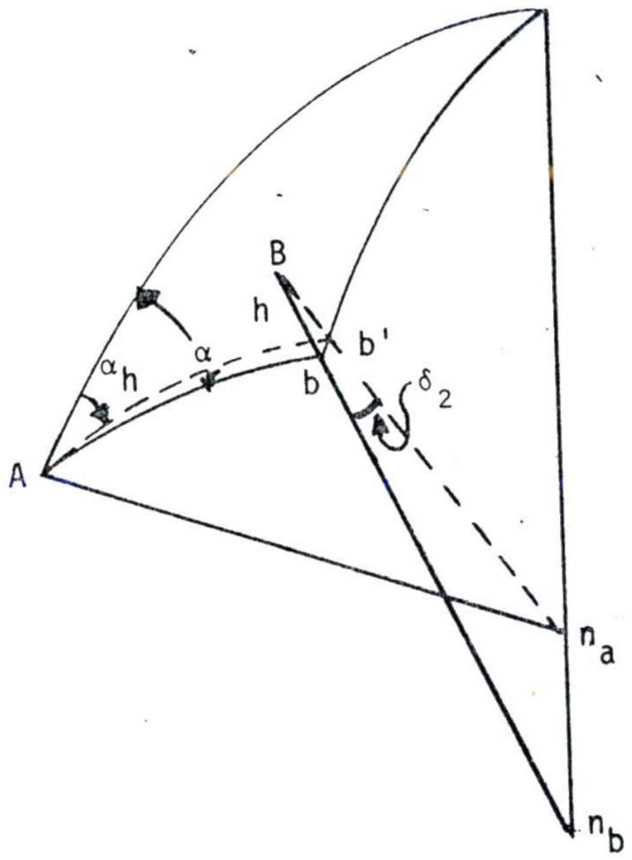
Geodetic azimuth from Hawagala to Kirioluhena - $138°02'50''.2$

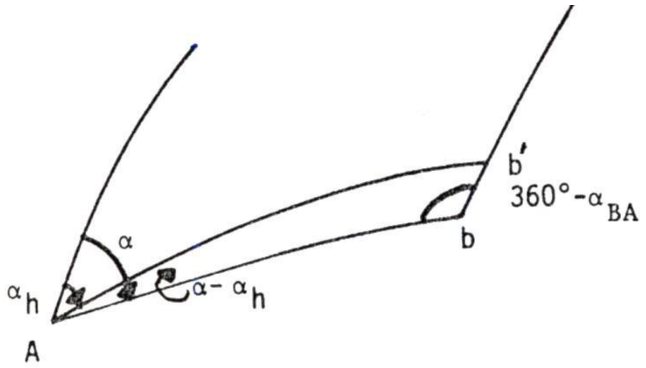
Ground distance from Hawagala to Kirioluhena -

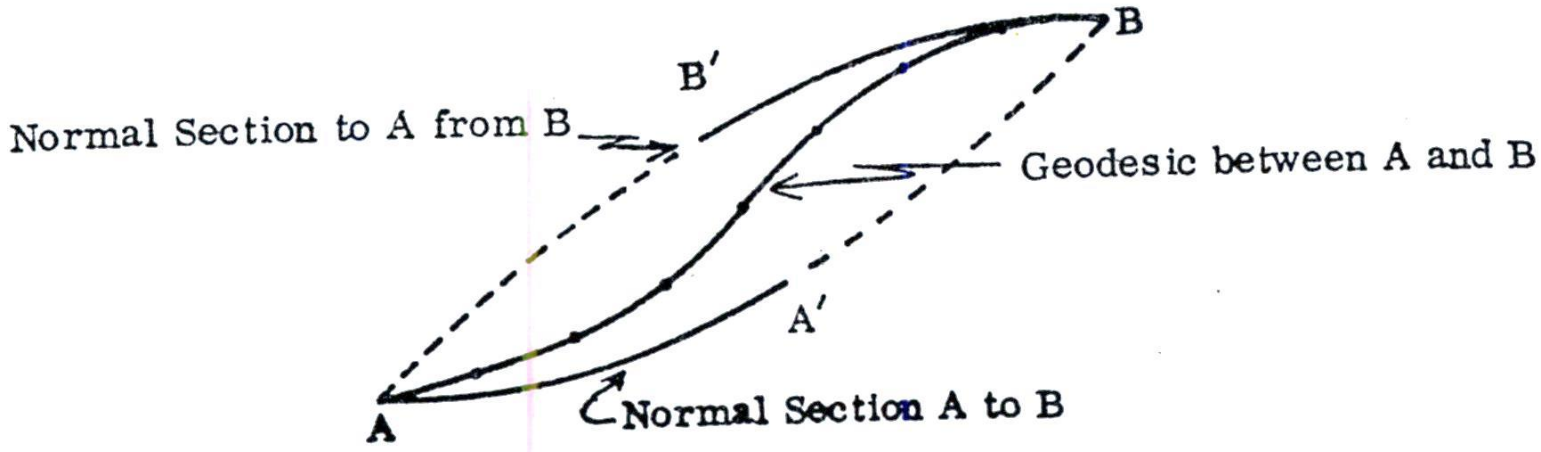
Calculate the geodetic latitudes and longitudes of stations ???

Computation Procedure

- Calculate mean angles and standard deviations
- Reduce observed directions and distances to ellipsoid
- Calculate approximate coordinates
  - Using the given azimuth and mean included angles, calculate azimuths of all other lines
  - Using Gauss Mid Latitude formula, calculate latitudes and longitudes of all unknown stations
- Using the approximate coordinates, calculate the approximate included angles (reverse of Mid latitude formula)and distances
- Form the F-matrix
- Using mean latitude and longitude for the entire area, calculate M and N
- Form B-matrix
- Solve the differentials and residuals. If residuals are too large, do iterations
- Calculate the corrected observed angles and coordinates of unknown stations


B
h
b'
α_h
α
b
δ_2
A
n_a
n_b


b'
$360° - \alpha_{BA}$
b
α
α_h
$\alpha - \alpha_h$
A


B
B'
Normal Section to A from B
Geodesic between A and B
A'
A
Normal Section A to B